(1,1)

$f_{xx}(x,y)=2y^3$, $f_{xy}(x,y)=f_{yx}(x,y)=6xy^2$

$f_{yy}(x,y)=6xy^2$

(1,2)

$(x,y)\neq(0,0)$で

$$f_{xy}(x,y)=\frac{x^2-y^2}{x^2+y^2}+\frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^3}$$

$\therefore f_{xy}(0,y)=-1$, $f_{xy}(x,0)=1$

$\therefore \lim_{y\to 0} f_{xy}(0,y)\neq\lim_{x\to 0} f_{xy}(x,0)$

よって $f_{xy}(x,y)$は原点近傍で不連続

ゆえに $f(x,y)$はC^2級でない //

(2,1)

部分系をiとし、$V^{(i)}=rV_{tot}$ ($r\in\mathbb{R}$, $0<r<1$)とする

rに収束する有理数列を$\{g_k\}$とし、

$g_k=\frac{b_k}{a_k}$ ($a_k, b_k\in\mathbb{N}$, $a_k\geq 1$)とする。

$V^{(k)}=g_kV_{tot}$, $V^{(l)}=\frac{1}{a_k}V_{tot}$となる部分系$k$, lを定めると

lはもとの系をa_k等分したものなのでもとの系の均一性より

$$X^{(l)}=X^{(l')}$$

が任意のl'(ただし$V^{(l')}=\frac{1}{a_k}V_{tot}$)に対して成立

$\therefore X_{tot}=\sum_{l=1}^{a_k}X^{(l)}=a_kX^{(l)}$ $\therefore X^{(l)}=\frac{1}{a_k}X_{tot}$

lは部分系kをb_k等分したものでもあるので

$$X^{(k)}=\sum_{l=1}^{b_k}X^{(l)}=b_kX^{(l)}=\frac{b_k}{a_k}X_{tot}=g_kX_{tot}$$

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$k \to \infty$として

$$X^{(2)} = r X_{tot} = \frac{V^{(2)}}{V_{tot}} X_{tot}$$

$K = \frac{X_{tot}}{V_{tot}}$とすると $X^{(2)} = K V^{(2)}$ //

(4.1) $f(x)$が上に凸なとき、$f''(x) \leq 0$ ―― ①
対称性より $a < b$ としても一般性を失わない。
このとき
$\lambda a + (1-\lambda) b > a$ ―― ②
$g(a,b) = f(\lambda a + (1-\lambda) b) - \lambda f(a) - (1-\lambda) f(b)$
とおくと

$$\begin{aligned} \frac{d}{da} g(a,b) &= \lambda f'(\lambda a + (1-\lambda) b) - \lambda f'(a) \\ &= \lambda \left(f'(\lambda a + (1-\lambda) b) - f'(a) \right) \\ &\leq 0 \quad (\because ① ②) \end{aligned}$$

よって $g(a,b)$ は a に対して単調減少
$\therefore g(a,b) \geq g(b,b) = 0$
$\therefore f(\lambda a + (1-\lambda) b) \geq \lambda f(a) + (1-\lambda) f(b)$ //

(4.2)
1.
$$\begin{aligned} f(\lambda a + (1-\lambda) b) &= -(\lambda a + (1-\lambda) b)^2 \\ &= -\lambda^2 a^2 - 2\lambda(1-\lambda) ab - (1-\lambda)^2 b^2 \\ &= -\lambda a^2 - (1-\lambda) b^2 + \lambda(1-\lambda)(a-b)^2 \\ &\geq \lambda f(a) + (1-\lambda) f(b) \end{aligned}$$

$\therefore$ 上に凸

2. $g(a,b) = f(\lambda a + (1-\lambda) b) - \lambda f(a) - (1-\lambda) f(b)$
とおく $(a > 0, b > 0)$

$$\frac{\partial}{\partial a} g(a,b) = \frac{\lambda}{\lambda a + (1-\lambda) b} - \frac{\lambda}{a} = \frac{\lambda(1-\lambda)(a-b)}{a(\lambda a + (1-\lambda) b)}$$

i) $a>b$ のとき $\frac{\partial}{\partial a} g(a,b)>0$

$\therefore g(a,b) > g(b,b) = 0$

ii) $a<b$ のとき $\frac{\partial}{\partial a} g(a,b)<0$

$g(a,b) > g(b,b) = 0$

i) ii) より $f(\lambda a+(1-\lambda)b) > \lambda f(a) + (1-\lambda)f(b)$

$\therefore$ 上に凸

3. $(f(\lambda a+(1-\lambda)b))^2 - (\lambda f(a) + (1-\lambda)f(b))^2$

$= \lambda(1-\lambda)(\sqrt{-a} - \sqrt{-b})^2$

≥ 0

これと $f(x) \geq 0$ より

$f(\lambda a+(1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b)$

$\therefore$ 上に凸

(4.3)

1. $h(\lambda a+(1-\lambda)b) = f(\lambda a+(1-\lambda)b) + g(\lambda a+(1-\lambda)b)$

$\geq \lambda(f(a)+g(a)) + (1-\lambda)(f(b)+g(b))$

$= \lambda h(a) + (1-\lambda) h(b)$

$\therefore$ 上に凸

2. $h(\lambda a+(1-\lambda)b) = f(X-\lambda a-(1-\lambda)b)$

$= f(\lambda(X-a)+(1-\lambda)(X-b))$

$\geq \lambda f(X-a) + (1-\lambda) f(X-b)$

$= \lambda h(a) + (1-\lambda) h(b)$

$\therefore$ 上に凸

3. $h(\lambda a+(1-\lambda)b) = f(\lambda a+(1-\lambda)b) + g(\lambda(X-a)+(1-\lambda)(X-b))$

$\geq \lambda(f(a)+g(X-a)) + (1-\lambda)(f(b)+g(X-b))$

$= \lambda h(a) + (1-\lambda) h(b)$ $\therefore$ 上に凸

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(4.4)

1. $h(\lambda\vec{a}+(1-\lambda)\vec{b})=f(\lambda\vec{a}+(1-\lambda)\vec{b})+g(\lambda\vec{a}+(1-\lambda)\vec{b})$
$\geq\lambda(f(\vec{a})+g(\vec{a}))+(1-\lambda)(f(\vec{b})+g(\vec{b}))$
$=\lambda h(\vec{a})+(1-\lambda)h(\vec{b})$

∴ 上に凸

2. $h(\lambda\vec{a}+(1-\lambda)\vec{b})=f(\vec{x}-\lambda\vec{a}-(1-\lambda)\vec{b})$
$=f(\lambda(\vec{x}-\vec{a})+(1-\lambda)(\vec{x}-\vec{b}))$
$\geq\lambda f(\vec{x}-\vec{a})+(1-\lambda)f(\vec{x}-\vec{b})$
$=\lambda h(\vec{a})+(1-\lambda)h(\vec{b})$

∴ 上に凸

3. $h(\lambda\vec{a}+(1-\lambda)\vec{b})=f(\lambda\vec{a}+(1-\lambda)\vec{b})+g(\lambda(\vec{x}-\vec{a})+(1-\lambda)(\vec{x}-\vec{b}))$
$\geq\lambda(f(\vec{a})+g(\vec{x}-\vec{a}))+(1-\lambda)(f(\vec{b})+g(\vec{x}-\vec{b}))$
$=\lambda h(\vec{a})+(1-\lambda)h(\vec{b})$

∴ 上に凸

(4.5)

1. $h_1(\lambda A+(1-\lambda)B,\lambda a+(1-\lambda)b)$
$=f(\lambda a+(1-\lambda)b)+f(\lambda(A-a)+(1-\lambda)(B-b))$
$\geq\lambda(f(a)+f(A-a))+(1-\lambda)(f(b)-f(B-b))$
$=\lambda h_1(A,a)+(1-\lambda)h_1(B,b)$

$h_2(\lambda A+(1-\lambda)B,\lambda a+(1-\lambda)b)$
$=f(\lambda(A+\frac{a}{2})+(1-\lambda)(B+\frac{b}{2}))+f(\lambda(A-\frac{a}{2})+(1-\lambda)(B-\frac{b}{2}))$
$\geq\lambda(f(A+\frac{a}{2})+g(A-\frac{a}{2}))+(1-\lambda)(f(B+\frac{b}{2})+f(B-\frac{b}{2}))$
$=\lambda h_2(A,a)+(1-\lambda)h_2(B,b)$

以上より $h_1(X,x)$, $h_2(X,x)$ は共に上に凸

2. Xを固定したとき

$$h_1'(X,x) = f'(x) - f'(X-x)$$
$$h_2'(X,x) = \frac{1}{2}\left(f'(X+\frac{x}{2}) - f'(X-\frac{x}{2})\right)$$

$$h_1'(X,x)=0 \Leftrightarrow f'(x)=f'(X-x) \Leftrightarrow x=X-x \Leftrightarrow x=\frac{1}{2}X$$
$$h_2'(X,x)=0 \Leftrightarrow X+\frac{x}{2}=X-\frac{x}{2} \Leftrightarrow x=0$$

($\because f'(x)$は単調減少)

よってh_1, h_2が最大になるのはそれぞれ$x=\frac{1}{2}X$, $x=0$のとき

(4.6)

$$S(\lambda U,\lambda V)=k_B\left[\left(\frac{\lambda U}{\varepsilon}+\frac{\lambda V}{\gamma}\right)\ln\left(\frac{\lambda U}{\varepsilon}+\frac{\lambda V}{\gamma}\right)-\frac{\lambda U}{\varepsilon}\ln\frac{\lambda U}{\varepsilon}-\frac{\lambda V}{\gamma}\ln\frac{\lambda V}{\gamma}\right]$$
$$=\lambda k_B\left[\left(\frac{U}{\varepsilon}+\frac{V}{\gamma}\right)\ln\left(\frac{U}{\varepsilon}+\frac{V}{\gamma}\right)-\frac{U}{\varepsilon}\ln\frac{U}{\varepsilon}-\frac{V}{\gamma}\ln\frac{V}{\gamma}\right]$$
$$=\lambda S(U,V)$$

$\therefore$ 1次同次関数

(4.7)

$$s = S/V$$
$$= k_B\left[\left(\frac{U}{\varepsilon V}+\frac{1}{\gamma}\right)\ln\left(\frac{U}{\varepsilon}+\frac{V}{\gamma}\right)-\frac{U}{\varepsilon V}\ln\frac{U}{\varepsilon}-\frac{1}{\gamma}\ln\frac{V}{\gamma}\right]$$
$$= k_B\left[\left(\frac{u}{\varepsilon}+\frac{1}{\gamma}\right)\ln\left(\frac{u}{\varepsilon}+\frac{1}{\gamma}\right)-\frac{u}{\varepsilon}\ln\frac{u}{\varepsilon}-\frac{1}{\gamma}\ln\frac{1}{\gamma}\right]$$

(4.8)

$$\frac{\partial S}{\partial U}=\frac{k_B}{\varepsilon}\left(\ln\left(\frac{U}{\varepsilon}+\frac{V}{\gamma}\right)-\ln\frac{U}{\varepsilon}\right)$$

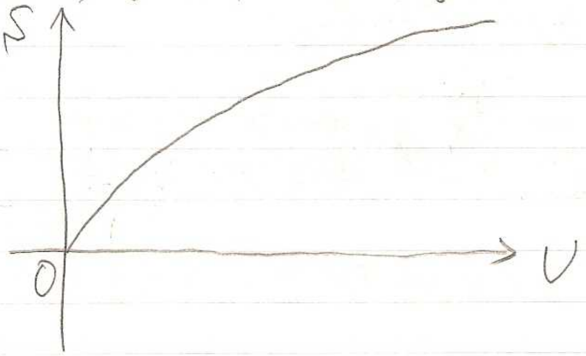
$$\frac{\partial^2 S}{\partial U^2} = -\frac{k_B V}{U(\gamma U+\varepsilon V)} < 0$$

$$\frac{\partial S}{\partial V}=\frac{k_B}{\gamma}\left(\ln\left(\frac{U}{\varepsilon}+\frac{V}{\gamma}\right)-\ln\frac{V}{\gamma}\right),\quad \frac{\partial^2 S}{\partial V^2}=-\frac{k_B U}{V(\gamma U+\varepsilon V)} < 0$$

よってSはU, Vについて上に凸

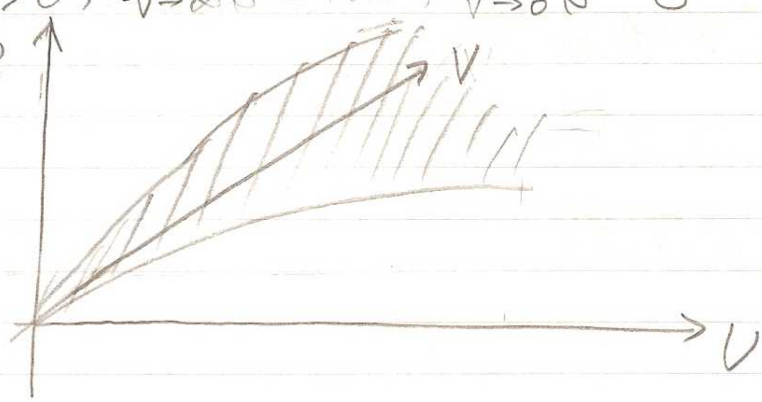
(4,9)

$\frac{\partial S}{\partial U} > 0$, $\lim_{U\to\infty} S = +\infty$, $\lim_{U\to 0} S = 0$



(4,10)

$\frac{\partial S}{\partial U} > 0$, $\lim_{U\to\infty} S = +\infty$, $\lim_{U\to 0} S = 0$



(4,11)

1. $\tilde{S} = S(U^{(1)}, \frac{V}{2}) + S(U^{(2)}, \frac{V}{2}) = S(U^{(1)}, \frac{V}{2}) + S(U-U^{(1)}, \frac{V}{2})$

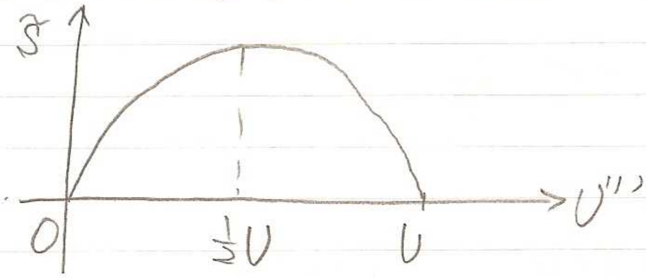
$$= k_B\left[\left(\frac{U^{(1)}}{\varepsilon} + \frac{V}{2\gamma}\right)\ln\left(\frac{U^{(1)}}{\varepsilon}+\frac{V}{2\gamma}\right) + \left(\frac{U-U^{(1)}}{\varepsilon}+\frac{V}{2\gamma}\right)\ln\left(\frac{U-U^{(1)}}{\varepsilon}+\frac{V}{2\gamma}\right)\right.$$
$$\left. - \frac{U^{(1)}}{\varepsilon}\ln\frac{U^{(1)}}{\varepsilon} - \frac{U-U^{(1)}}{\varepsilon}\ln\frac{U-U^{(1)}}{\varepsilon} - \frac{V}{\gamma}\ln\frac{V}{2\gamma}\right]$$

2. $f(U) = \frac{\partial}{\partial U} S(U, \frac{V}{2})$ とおく。$f'(U) = \frac{\partial^2}{\partial U^2} S(U, \frac{V}{2}) > 0$

$$\frac{\partial}{\partial U^{(1)}}\tilde{S} = f(U^{(1)}) - f(U-U^{(1)})$$

$\frac{\partial}{\partial U^{(1)}} \tilde{S} = 0$ のとき $U^{(1)} = U - U^{(1)}$ ∴ $U^{(1)} = \frac{1}{2}U$

$$\frac{\partial^2}{\partial U^{(1)2}} \tilde{S} = f''(U^{(1)}) + f''(U - U^{(1)}) > 0$$



3. $U^{(1)} = \frac{1}{2}U$ で最大値

$$\max \tilde{S} = 2S\left(\tfrac{1}{2}U, \tfrac{1}{2}V\right) = S(U, V) \quad (\because S\text{は1次同次関数})$$

$$= k_B\left[\left(\frac{U}{\varepsilon} + \frac{V}{\gamma}\right)\ln\left(\frac{U}{\varepsilon} + \frac{V}{\gamma}\right) - \frac{U}{\varepsilon}\ln\frac{U}{\varepsilon} - \frac{V}{\gamma}\ln\frac{V}{\gamma}\right]$$

(5, 1)

1. $B = \frac{\partial S}{\partial U} = \frac{k_B}{\varepsilon}\left(\ln\left(\frac{U}{\varepsilon} + \frac{V}{\gamma}\right) - \ln\frac{U}{\varepsilon}\right)$

2. $\Pi_V = \frac{\partial S}{\partial V} = \frac{k_B}{\gamma}\left(\ln\left(\frac{U}{\varepsilon} + \frac{V}{\gamma}\right) - \ln\frac{V}{\gamma}\right)$

(5, 2)

$$\Pi_k = \frac{\partial S}{\partial X_k} = \frac{\partial}{\partial X_k} V s(u, n, \cdots, x_k, \cdots, x_t) = \frac{\partial}{\partial X_k} V s\left(u, n, \cdots, \frac{X_k}{V}, \cdots, x_t\right)$$
$$= \frac{\partial}{\partial x_k} s(u, n, \cdots, x_k, \cdots, x_t)$$

$$P_k = \frac{\partial U}{\partial X_k} = \frac{\partial}{\partial X_k} V u\left(x_0, x_1, \cdots, \frac{X_k}{V}, \cdots, x_t\right) = \frac{\partial}{\partial x_k} u(x_0, \cdots, x_k, \cdots, x_t)$$

(5, 3)

1. $\frac{k_B}{\varepsilon}\left(\ln\left(\frac{U}{\varepsilon} + \frac{V}{\gamma}\right) - \ln\frac{U}{\varepsilon}\right) = \frac{1}{T}$

$$\frac{\varepsilon}{k_B T} = \ln\left(1 + \frac{\varepsilon V}{\gamma U}\right) \quad \therefore U = \frac{\varepsilon V}{\gamma\left(\exp\left(\frac{\varepsilon}{k_B T}\right) - 1\right)}$$

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2. $P = T\Pi_V = \frac{k_BT}{\gamma}\left(\ln\left(\frac{U}{\varepsilon}+\frac{V}{\gamma}\right) - \ln\frac{V}{\gamma}\right)$

$= \frac{k_BT}{\gamma}\left(\frac{\varepsilon}{k_BT} - \ln\left(\exp\left(\frac{\varepsilon}{k_BT}\right)-1\right)\right)$

$= \frac{\varepsilon}{\gamma} - \frac{k_BT}{\gamma}\ln\left(\exp\left(\frac{\varepsilon}{k_BT}\right)-1\right)$ //

3. $k_BT \gg \varepsilon \iff \frac{\varepsilon}{k_BT} \ll 1$

$\therefore U \simeq \frac{\varepsilon V}{\gamma \frac{\varepsilon}{k_BT}} = \frac{k_B}{\gamma} VT \quad \therefore U \propto V,\ U \propto T$

(5.4)

$S(\lambda U, \lambda V, \lambda N) = \frac{\lambda N}{N_0} S_0 + R\lambda N \ln\left[\left(\frac{\lambda U}{U_0}\right)^c \left(\frac{\lambda V}{V_0}\right)\left(\frac{N_0}{\lambda N}\right)^{c+1}\right]$

$= \lambda\left(\frac{N}{N_0}S_0 + RN\ln\left[\left(\frac{U}{U_0}\right)^c\left(\frac{V}{V_0}\right)\left(\frac{N_0}{N}\right)^{c+1}\right]\right.$

$= \lambda S$

$\therefore$ 1次同次関数

(5.5)

$s = S/V = \frac{N}{N_0 V} S_0 + R\frac{N}{V}\ln\left[\left(\frac{U}{U_0 V}\right)^c\left(\frac{1}{V_0}\right)\left(\frac{VN_0}{N}\right)^{c+1}\right]$

$= \frac{n}{N_0}S_0 + Rn\ln\left[\left(\frac{u}{U_0}\right)^c \cdot \frac{1}{V_0}\left(\frac{N_0}{n}\right)^{c+1}\right]$

(5.6)

$S = \frac{S_0}{N_0}N + RN\left(c\ln\frac{U}{U_0} + \ln\frac{V}{V_0} + (c+1)\ln\frac{N_0}{N}\right)$

$\frac{\partial S}{\partial U} = \frac{cRN}{U} \quad \frac{\partial^2 S}{\partial U^2} = -\frac{cRN}{U^2} < 0,\ \frac{\partial S}{\partial V} = \frac{RN}{V},\ \frac{\partial^2 S}{\partial V^2} = -\frac{RN}{V^2} < 0$

$\frac{\partial S}{\partial N} = \frac{S_0}{N_0} + R\left(c\ln\frac{U}{U_0} + \ln\frac{V}{V_0} + (c+1)\ln\frac{N_0}{N}\right) - (c+1)R$

$\frac{\partial^2 S}{\partial N^2} = -\frac{(c+1)R}{N} < 0$ よってSはU, V, Nについて上に凸

(5.7)

$$\mu = \frac{\partial U}{\partial N} = U_0 \left(\frac{N}{N_0}\right)^{\frac{1}{c}} \left(\frac{V_0}{V}\right)^{\frac{1}{c}} \frac{(c+1)RN - N_0 S}{cRNN_0} \exp\left[\frac{1}{cR}\left(\frac{S}{N} - \frac{S_0}{N_0}\right)\right]$$

$$T = \frac{\partial U}{\partial S} = \frac{U_0}{cRN} \left(\frac{N}{N_0}\right)^{\frac{c+1}{c}} \left(\frac{V_0}{V}\right)^{\frac{1}{c}} \exp\left[\frac{1}{cR}\left(\frac{S}{N} - \frac{S_0}{N_0}\right)\right]$$

$$P = -\frac{\partial U}{\partial V} = \frac{U_0}{cV} \left(\frac{N}{N_0}\right)^{\frac{c+1}{c}} \left(\frac{V_0}{V}\right)^{\frac{1}{c}} \exp\left[\frac{1}{cR}\left(\frac{S}{N} - \frac{S_0}{N_0}\right)\right]$$

(5.8)

$$cRNT = U_0 \left(\frac{N}{N_0}\right)^{\frac{c+1}{c}} \left(\frac{V_0}{V}\right)^{\frac{1}{c}} \exp\left[\frac{1}{cR}\left(\frac{S}{N} - \frac{S_0}{N_0}\right)\right] = U$$

$$PV = \frac{1}{c} U_0 \left(\frac{N}{N_0}\right)^{\frac{c+1}{c}} \left(\frac{V_0}{V}\right)^{\frac{1}{c}} \exp\left[\frac{1}{cR}\left(\frac{S}{N} - \frac{S_0}{N_0}\right)\right] = \frac{U}{c} = RNT$$

(5.9)

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k_B}{\varepsilon}\left[\ln\left(\frac{U}{\varepsilon} + \frac{V}{\gamma}\right) - \ln\frac{U}{\varepsilon}\right]$$

$$U = \frac{\varepsilon V}{\gamma\left(\exp\left(\frac{\varepsilon}{k_B T}\right) - 1\right)}$$

代入して

$$S = \frac{k_B V}{\gamma}\left[\frac{\varepsilon \exp\left(\frac{\varepsilon}{k_B T}\right)}{k_B T\left(\exp\left(\frac{\varepsilon}{k_B T}\right) - 1\right)} - \ln\left(\exp\left(\frac{\varepsilon}{k_B T}\right) - 1\right)\right]$$

$$= \frac{k_B V}{\gamma}\left[\frac{\varepsilon}{k_B T}\left(\frac{\exp\left(\frac{\varepsilon}{k_B T}\right)}{\exp\left(\frac{\varepsilon}{k_B T}\right) - 1} - 1\right) + \frac{\varepsilon}{k_B T} - \ln\left(\exp\left(\frac{\varepsilon}{k_B T}\right) - 1\right)\right]$$

$$= \frac{k_B V}{\gamma}\left[\frac{\varepsilon}{k_B T\left(\exp\left(\frac{\varepsilon}{k_B T}\right) - 1\right)} + \ln\frac{1}{1 - \exp\left(-\frac{\varepsilon}{k_B T}\right)}\right]$$

$$\longrightarrow 0 \quad (T \to +0)$$

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(5,10)
$$S=\frac{N}{N_0}S_0+RNc\ln U+(U\text{を含まない式})$$

$$\frac{1}{T}=\frac{\partial S}{\partial U}=\frac{cRN}{U}\quad\therefore U=cRNT\quad\therefore\lim_{T\to+0}U=0$$

$$\therefore\lim_{T\to+0}S=\lim_{U\to 0}S=-\infty\quad\therefore\text{不成立}$$

(6.1)
$$P=-\frac{\partial U}{\partial V}=\frac{1}{c}U_0\left(\frac{N}{N_0}\right)^{\frac{c+1}{c}}V_0^{\frac{1}{c}}V^{-\frac{c+1}{c}}\exp\left[\frac{1}{cR}\left(\frac{S}{N}-\frac{S_0}{N_0}\right)\right]$$

$$W_M=-\int_{V_1}^{V_2}PdV=U_0\left(\frac{N_0}{N}\right)^{\frac{c+1}{c}}\left[\left(\frac{V_0}{V_2}\right)^{\frac{1}{c}}-\left(\frac{V_0}{V_1}\right)^{\frac{1}{c}}\right]\exp\left[\frac{1}{cR}\left(\frac{S}{N}-\frac{S_0}{N_0}\right)\right]$$

(9.1)
$$dS_H+dS_L>0\quad\therefore -dS_H<dS_L\quad —①$$
$$d'Q=T_LdS_L=-T_HdS_H\quad —②$$
$d'Q>0$ とすると $dS_L>0$ よって①より

$$\frac{-dS_H}{dS_L}<1$$

②より $1>\frac{-dS_H}{dS_L}=\frac{T_L}{T_H}\quad\therefore T_H>T_L$

(10.1) 略
$U=cRNT$, $PV=NRT$, $\Delta U=W+Q$
だけでできるはず。(単原子理想気体では $c=\frac{3}{2}$)

(10.2) $Q=Q_L$
$W^{(e)}=W_1^{(e)}+W_2^{(e)}$ とすればよい

(10.3)
1. $\eta_{冷}\leq\frac{T}{T_0-T}=+\infty$